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Properly colored paths and cycles

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ABSTRACT

In an edge-colored graph, let $d^c(v)$ be the number of colors on the edges incident to v and let $\delta^c(G)$ be the minimum $d^c(v)$ over all vertices $v \in G$. In this work, we consider sharp conditions on $\delta^c(G)$ which imply the existence of properly edge-colored paths and cycles, meaning no two consecutive edges have the same color.

1. Introduction

Consider a network of relay stations in which there is a prescribed frequency channel available for message transmission between some pairs. If a message enters a relay station using a certain frequency, that station will be unable to reuse the frequency to send the message for fear of interference. This means that we do not allow two transmissions of the same frequency in a row.

If we translate the stations to vertices and connections to edges, we may label the edges of a graph with colors based on the frequencies. It is then natural to ask if, for every pair of vertices u and v in this graph, there is a path from u to v which is properly colored (i.e. no two consecutive edges of the same color).

Aside from the above application, properly colored paths and cycles appear in a variety of other fields including genetics [9–11] and social sciences [7]. There is also a good survey [1] dealing with the case where two colors are used on the edges. More recently, there has also been another survey of the area in Chapter 16 of [2]. The aforementioned chapter also includes a result (see Theorem 16.9.1) which may be helpful in proving some of the conjectures we pose in this paper. This result provides a characterization of colorings of complete graphs which contain a spanning properly colored path in terms of other spanning properly colored structures.

Much work has been done in finding properly colored subgraphs using conditions on the monochromatic degrees of vertices (number of incident edges of the same color) but, since we consider conditions on the number of *different* colors at vertices in this paper, we will not dwell on such results.

Throughout this work, we consider only edge colorings of simple graphs. Unless otherwise stated, the order of a graph G is $|G| = n$. Let the color degree $d^c_G(v)$ be the number of colors on edges incident to the vertex v in an edge-colored graph G. If the graph G is understood, we will simply use the notation $d^c(v)$. Let $\delta^c(G)$ be the minimum color degree of G. In [12], the authors considered what they called "alternating cycles" in edge-colored graphs, cycles in which no two adjacent edges share a color. Since this terminology can be a bit misleading when the number of colors is greater than 2, we prefer to call such cycles properly colored. For the ease of notation, we frequently call a subgraph proper in place of properly colored.

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In light of the classical theorem of Dirac [8] which states that if $\delta(G) \geq \frac{n}{2}$, then G is hamiltonian, the authors of [12] state the following conjecture. A graph will be called *properly hamiltonian* if it contains a properly colored hamiltonian cycle.

Conjecture 1 (*[12]*). *For $n \geq 3$, if $\delta^c(G) \geq \frac{n}{2}$, then G is properly hamiltonian.*

In [12], the authors showed the following partial result which will be useful in our proofs.

Theorem 1. *For $n \geq 4$, if $\delta^c(G) \geq \frac{n}{2}$ then G contains a properly colored cycle of length at least $\lceil \frac{n}{3} \rceil + 1$.*

The first main result of this work (Proposition 1) is a construction which shows that Conjecture 1 is not true as stated. The following examples will be used for sharpness throughout this work. This construction uses the fact that any odd-ordered complete graph K_{2n+1} can be decomposed into n edge-disjoint hamiltonian cycles which can be found in any beginning text on graph theory, see Theorem 9.21 in [6] for example.

Construction 1. *Consider a complete graph $G = K_{2m}$ let x be one of the vertices. Label the remaining vertices with $v_1, v_2, \ldots, v_{2m-1}$. Color the edge xv_i with color i for all i. Let $H = G \setminus x$ and arbitrarily partition $E(H)$ into $m-1$ hamiltonian cycles. Also, we arbitrarily orient these hamiltonian cycles. Color the edge v_iv_j with color j if the arc $\overrightarrow{v_iv_j}$ is an arc of one of the hamiltonian cycles. This provides a coloring of $E(G)$. Call this colored graph G_0. It will be shown that x is not contained in any properly colored cycle. Let $G_0' = G_0 \setminus v_1$.*

Let $G_1 = G_0 \cup y$ where y has an edge of color i to v_i for all i and an edge of color 1 to x. It will be shown that the vertices x and y have only one properly colored path between them, namely the edge xy. Furthermore, define $G_2 = G_1 \setminus e$ where $e = xy$. In this graph, there is no properly colored path from x to y. Finally, let $G_2' = G_2 - v_1$.

Proposition 1. *There exists a coloring G of K_{2m} with $\delta^c(G) = m$ which has no properly colored hamiltonian cycle.*

Proof. Consider the colored graph G_0 from Construction 1. Each vertex except x is contained in exactly $m-1$ of these cycles so each vertex v_i has $m-1$ out-edges. This means that $d^c(v_i) = (m-1) + 1 = m$ since v_i has an edge of color i to x and $m-1$ other colors on the out-edges of the cycles. Hence, $\delta^c(G_0) = \frac{|G_0|}{2}$.

In order to see that G_0 has no properly colored hamiltonian cycle, consider a properly colored path P beginning at x. Certainly the first edge of P, from x to v_i, must have color i. Since P may not use another edge of color i, it must leave v_i to another vertex v_j in a color other than i. By construction, this edge must have color j (and must follow the direction of one of the hamiltonian cycles) so P must enter v_j with color j. This means that the path P must always enter each vertex v_j with color j. Hence, G_0 contains no properly colored cycle containing x. In particular, G_0 contains no properly colored hamiltonian cycle. $\square$

A similar conjecture may still be true when $\delta^c(G) \geq \frac{n+1}{2}$.

Conjecture 2. *For $n \geq 3$, if $\delta^c(G) \geq \frac{n+1}{2}$, then G is properly hamiltonian.*

Proving Conjecture 2 seems difficult even for the case when the graph is complete. In Section 2, we consider properly colored cycles in complete graphs and propose the following conjecture. If $\delta^c(K_n) \geq \frac{n+1}{2}$ then we believe this colored complete graph is properly vertex pancyclic meaning that every vertex is contained in a properly colored cycle of all possible lengths. In support of this conjecture, we show the following.

Theorem 2. *If $n \geq 13$ and $\delta^c(K_n) \geq \frac{n+1}{2}$, then each vertex is contained in a proper triangle, a proper C_4 and a proper cycle C with $|C| \geq 5$.*

In trying to prove Conjecture 2, we consider a necessary condition for a graph to be properly hamiltonian, namely, it must be properly connected meaning that, between each pair of vertices, there exists a properly colored path. Unlike the standard minimum degree condition for a graph to be connected, it is far from trivial to show that a graph is properly connected. In Section 3, we determine the sharp minimum color degree condition ($\delta^c(G) \geq \frac{n}{2}$) for a graph to be properly connected. The results contained in this work could be a big step toward proving Conjecture 2.

We conclude this section with some definitions and an easy but useful observation. A graph is vertex pancyclic if each vertex is contained in a cycle of every possible length. A colored graph is said to be *properly k-connected* if for every pair of vertices u and v, there exist k internally disjoint properly colored paths from u to v. This definition is similar to the definition of rainbow connection [3–5]. With this in mind, we also say that a graph is *rainbow k-connected* if, between every pair of vertices, there exist k internally disjoint rainbow colored paths, meaning that each edge on a path has a different color. In an edge-colored complete graph, a set of vertices A is said to have the *dependence property* with respect to a vertex $v \notin A$ (denoted DP_v) if $c(aa') \in \{c(va), c(va')\}$ for all $a, a' \in A$. Using this property, one may easily obtain the following fact.

Fact 1. *If a set A has DP_v, then there exists a vertex $a \in A$ with*

1. *$d_A^c(a) \leq \frac{|A|+1}{2}$ and,*
2. *if $|A| \geq 2$, then at least one of the colors used in A at a is $c(va)$.*

Proof. For Item 1, suppose $|A| = m$ and orient the edges within A based on their colors so that if $c(xy) = c(xv)$, we direct the edge from x to y for all $x, y \in A$. If xv and yv have the same color, arbitrarily orient the edge. Thus, A becomes a tournament and every tournament has maximum out-degree at least $\frac{m+1}{2}$. Let u be a vertex of maximum out-degree in the tournament and note that this means u has at least $\frac{m+1}{2}$ incident edges in the color $c(uv)$. Therefore, u has at most $m - \frac{m+1}{2} + 1 = \frac{m+1}{2}$ different colors on incident edges.

For Item 2, the claim is easy to check when $|A| = 2$ so assume it is true when $|A| \leq m - 1$ and suppose $|A| = m$. Applying Item 1, there is a vertex a with $d_A^c(a) \leq \frac{m+1}{2}$ so, suppose there is no edge of color $c(va)$ incident to a in A. By induction, there exists a vertex a' in $A \setminus a$ with $d_{A\setminus a}^c(a') \leq \frac{m}{2}$ with an edge of color $c(va')$ incident to a' in $A \setminus a$. By the assumption on a, we know $c(aa') \neq c(va)$ which means that $c(aa') = c(va')$ (a color already used at a' in $A \setminus a$). Therefore $d_A^c(a') \leq \frac{m+1}{2}$ and a' is the desired vertex. □

2. Properly colored cycles

Within edge-colored complete graphs, we believe the following is true.

Conjecture 3. *If $\delta^c(K_n) \geq \frac{n+1}{2}$ then this colored complete graph is properly vertex pancyclic.*

A similar result may also be true for non-complete graphs, which would imply Conjecture 2. Although we were unable to show that a colored complete graph with color degree at least $\frac{n+1}{2}$ has a properly colored hamiltonian cycle, we can show the following results.

Theorem 3. *If $n \geq 3$ and $\delta^c(K_n) \geq \frac{n+1}{2}$ then every vertex is contained in a rainbow triangle.*

Proof. Let G be a coloring of K_n with $\delta^c(G) > \frac{n}{2}$. Consider an arbitrary vertex v, supposing that v is not in a rainbow triangle, and let $t = d^c(v)$. Let A_i be the set of vertices with edges of color i to v for all $1 \leq i \leq t$. Note that all edges between the sets A_i and A_j must be either color i or color j for all $i \neq j$ to avoid a rainbow triangle at v, meaning that $G \setminus v$ has DP_v. Suppose $|A_1| = |A_2| = \cdots = |A_r| = 1$ and $2 \leq |A_{r+1}| \leq \cdots \leq |A_t|$ and let $A = G[A_1 \cup \ldots \cup A_r]$ and $H = G[A_{r+1} \cup \cdots \cup A_t]$ and set $s = t - r$.

Certainly $r \geq 2$ since $d^c(v) > \frac{n}{2}$ but we will first suppose that $r \geq 4$. By Fact 1, there exists a vertex $u \in A$ with $d_A^c(u) \leq \frac{r+1}{2}$ and one of these colors is $c(vu)$. Since $A \cup H$ has DP_v and $|A_i| \geq 2$ for all $i > r$, this means that $d^c(u) \leq \frac{r+1}{2} + s \leq \frac{|A|+1}{2} + \frac{|H|}{2} \leq \frac{n}{2}$, a contradiction.

Finally we suppose that $2 \leq r \leq 3$. Certainly there exists a vertex $u \in A$ with an edge (within A) of color $c(uv)$. Then $d^c(u) \leq r - 1 + s \leq \frac{n}{2}$, a contradiction completing the proof. □

Using Theorem 3, we also get the following which places each vertex on a properly colored C_4.

Theorem 4. *If $n \geq 4$ and $\delta^c(K_n) \geq \frac{n+1}{2}$ then every vertex is contained in a properly colored C_4.*

Proof. If $n = 4$, the color degree of each vertex is 3 so every C_4 is properly colored. If $n = 5$ or 6, from the color degree condition, each vertex has at most two incident edges with the same color. Thus, if $n = 6$, we may simply remove any vertex v and reduce to the case where $n = 5$. Note that a different vertex, other than v, can be removed to show that v is also contained in a proper C_4. There are 15 different copies of C_4 as subgraphs of K_5 and, for a contradiction, each one must contain two consecutive edges of the same color. Each pair of edges incident to a single vertex is used in exactly two copies of C_4 so, since each vertex has at most two incident edges with the same color, at most 10 copies of C_4 contain such edges, meaning that at least 5 copies of C_4 are properly colored. Thus, each vertex is contained in at least one of these copies of C_4.

Now suppose $n = 7$ or 8 and again we reduce to the case where $n = 7$ by simply removing any vertex. Choose a vertex v and we would like to show that v is contained in a properly colored C_4. By Theorem 3, v lies in a proper triangle $T = vxy$ so suppose $c(vx) = 1$, $c(vy) = 2$ and $c(xy) = 3$. By the color degree assumption, each of v, x and y are incident to two edges which are not the same colors as their incident edges in T. Since there are only 4 vertices outside T, there must be a vertex w which is incident to two of these edges. Under these conditions, we may assume $c(xw) = 4$ and $c(yw) = 4$ since all other cases may be solved by an identical argument.

In order to avoid a proper C_4 using $vxywv$ or $vyxwv$, the edge vw must also have color 4. So far, w has three incident edges all of color 4 so, by the degree assumption, all other edges incident to w must have distinct colors other than 4. One such edge, say to a vertex u, must have a color other than 1 or 2, say i. Then to avoid a proper C_4 using $vxwuv$ or $vywuv$, the edge uv must have color i. To avoid a proper C_4 using $xuwvx$ or $xuvyx$, the edge xu must have color i and with a symmetric argument, yu must also have color i. This means that u has 4 incident edges in color i, a contradiction to the color degree assumption, completing the proof when $n = 7$ or 8.

Now suppose $n \geq 9$. Let G be a coloring of K_n with $\delta^c(G) \geq \frac{n+1}{2}$ and let $v \in G$. By Theorem 3, v is contained in a rainbow triangle T. Label the vertices of T with $\{u, v, w\}$ and suppose $c(uv) = 1$, $c(vw) = 2$ and $c(uw) = 3$. The remainder of the proof is broken into two cases.

Case 1. *There exists a vertex $x \in G \setminus T$ such that $c(vx) > 3$ (suppose color 4) and $c(ux) = c(wx) = 3$.*

Let $Q = T \cup x$. Let $y \in G \setminus Q$ be a vertex with $c(xy) = c \geq 5$. Since $n \geq 9$, there must exist such a vertex. In order to avoid a proper C_4 on $yxuvy$, we know $c(vy)$ is either c or 1. On the other hand, in order to avoid a proper C_4 on $yxwvy$, we know $c(vy)$ is either c or 2. This means that $c(vy) = c(xy) = c$. Similarly, using the potential cycles $ywuvy$ and $ywvxy$, we see that $c(yw)$ must also be c. By symmetry, we get $c(yu) = c$. This means that every vertex with an edge of color $c \geq 5$ to x must also have edges of color c to every vertex of Q.

Let A_i be the set of vertices $a \in G \setminus Q$ such that $c(ax) = i$ for all $i \geq 5$ and let $A = \cup A_i$. Suppose there are t such sets. Since each set A_i has all the same colored edges to Q for all i, A must have DP_v to avoid creating a proper C_4 using $aa'wva$ for some pair $a, a' \in A$. By Fact 1, there exists $a \in A$ with $d_A^c(a) \leq \frac{t+1}{2}$.

Now suppose a vertex $y \in S = G \setminus (Q \cup A)$ has an edge of color $c > 5$ to v. Note that $n \geq |S| + t + 4$. In order to avoid a proper C_4 using $yvuxy$, the edge yx must be either color 3 or c. If it has color c, then $y \in A$ so suppose $c(xy) = 3$. Consider a vertex $a \in A_i$. To avoid a proper C_4 using $ayxva$, the edge ay must have either color i or color 3. This means that ay either has color i or color 3 for all $y \in S$ and $a \in A$.

Let $y \in S$ and $a \in A_i$. In order to avoid a properly colored C_4 using $yawvy$ or $yauvy$, the edge ya must either have color i or one of the colors in $\{1, 2, 3\}$. Hence, a vertex $a \in A$ can have at most 3 colors other than color i on edges to S so set $s = \min\{|S|, 3\}$. This implies the following inequality.

$$\frac{t+1}{2} + s \geq \frac{n+1}{2}.$$

Simplifying this inequality yields $t + 2s \geq n$ but this is a contradiction because $n \geq s + t + 4$ and $s \leq 3$, completing the proof in this case.

Case 2. *There exists no vertex $x \in G \setminus T$ with $c(vx) > 3$ and $c(ux) = c(wx) = 3$.*

Suppose $x \in G \setminus T$ with $c(xv) = c > 2$. Since $n \geq 5$, such a vertex must exist. In order to avoid a proper C_4 on $xvuwx$, the edge wx must have either color 3 or c (note that if $c = 3$, the edge must have color 3). If this edge has color 3, then to avoid a proper C_4 using $xuvwx$ or $xuwvx$, we get $c(ux) = 3$ and, when $c > 3$, this contradicts the assumption of this case. This means that $c(wx) = c$ and symmetrically $c(ux) = c$. Hence, for any vertex $x \in G \setminus T$, if $c(vx) = c > 2$ then x has edges of color c to all the vertices of T.

Let A_i be the set of vertices with edges of color i to v for $i > 2$ and let $A = \cup A_i$. Let $S = G \setminus (A \cup T)$. Suppose a vertex $y \in S$ has an edge of color $c \neq i$ to a vertex $a \in A_i$ for some i such that $c > 2$. Then $c(yv)$ must be 1 to avoid a proper C_4 on $ayvua$ but $c(yv)$ must be 2 to avoid a proper C_4 on $ayvwa$, a contradiction. Therefore, all edges from a vertex $a \in A_i$ to S must have either color i or a color from the set $\{1, 2\}$. Set $s = \min\{|S|, 2\}$.

The set A has DP_v since otherwise va_ia_juv is a proper C_4 containing v where a_ia_j is a pair violating DP_v. Let a be a vertex of minimum color degree within $G \setminus S$. Note that, by Fact 1, $d^c(a) \leq \frac{|A|+1}{2}$. The vertex a can also gain at most s more colors to S so, by the degree condition, we get the following.

$$\frac{|A|+1}{2} + s \geq \frac{n+1}{2}.$$

Simplifying this yields $|A| + 2s \geq n$. This contradicts $n \geq s + |A| + 3$ because $0 \leq s \leq 2$. □

For the sharpness of this result, we again use the graph G_0 from Construction 1. As shown in the proof of Proposition 1, this graph has $\delta^c(G_0) = \frac{n}{2}$ but no proper cycle containing the vertex x. For the case when n is odd, the graph G_0' has $\delta^c(G_0) = \frac{n-1}{2}$ and still no proper cycle containing x.

Theorem 5. *If $n \geq 13$ and $\delta^c(K_n) \geq \frac{n+1}{2}$ then every vertex v is contained in a proper cycle of length at least 5.*

Proof. Let G be a coloring of K_n with $\delta^c(G) \geq \frac{n+1}{2}$ and let $v \in G$. By Theorem 1, there exists a longest properly colored cycle C of length $\ell > \frac{n}{3}$. Let $C = v_1v_2, \ldots, v_1$. If v is on this cycle, the proof is done since $n \geq 13$ meaning that $|C| = \ell \geq \lceil \frac{13}{3} \rceil + 1 = 6$, so suppose $v \in G \setminus C$. We break the remainder of the proof into cases based on the colors of the edges between v and C. For the remainder of this proof, all indices will be modulo ℓ. For the sake of the case division, we say that a vertex $x \in G \setminus C$ *follows the colors of C increasing* if $c(xv_i) = c(v_iv_{i+1})$ (or symmetrically we say that x follows the colors of C *decreasing* if $c(xv_i) = c(v_iv_{i-1})$) for all i.

Case 1. *v has only one color on all edges to C.*

Suppose $c(vv_i) = 1$ for all i. Since $\delta^c(G) \geq \frac{n+1}{2}$, there exist at least $\frac{n-1}{2}$ vertices in $G \setminus C$ with edges of distinct colors (other than color 1) to v. Let $u \in G \setminus (C \cup v)$ and suppose $c(uv) = 2$. If $c(uv_i) \neq 2$ for some i, then either $C' = v_iuvv_{i+1}Cv_i$ or $C'' = v_iuvv_{i-1}Cv_i$ is a long properly colored cycle containing v. This means that $c(uv_i) = 2$ for all i and this is true for all $u \in G \setminus (C \cup v)$ with $c(uv) \neq 1$. Let $D = \{u \in G \setminus (C \cup v) : c(uv) \neq 1\}$ and recall that $|D| \geq \frac{n-1}{2}$.

All vertices of D have color degree $\frac{n+1}{2}$ within $G \setminus C$ (note that this set contains v) so we will focus on $G \setminus C$ for the remainder of this case. We know that $n' = |G \setminus C| \leq \frac{2n}{3}$ so the vertices of D have color degree at least $\frac{3n'}{4}$ in $G \setminus C$. We may easily construct a properly colored P_4, call it P, with v at one end and otherwise using only vertices of D. Let x be the opposite end of P (from v) and let c and c' be the colors of the edges of P incident to v and x respectively. Let $X \subseteq G \setminus (C \cup P)$ be the set of vertices with edges of colors other than c' to x. Note that $|X| \geq \frac{3n'}{4} - 1$. Similarly let $V \subseteq G \setminus (C \cup P)$ be the set of vertices with edges of colors other than c to v and again $|V| \geq \frac{3n'}{4} - 1$. Note that $|X \cap V| > \frac{n'}{2}$ (and of course $|X \cap V| \leq n' - 4$).

For every $w \in X \cap V$, we have $c(vw) = c(wx)$ to avoid a properly colored C_5 containing v. Furthermore, to avoid a proper C_6 containing v, each edge w_1w_2 for $w_i \in X \cap V$ must have either $c(w_1w_2) = c(vw_1)$ or $c(w_1w_2) = c(vw_2)$ so $X \cap V$ has DP_v. By Fact 1, there exists a vertex $y \in X \cap V$ with $d^c_{X\cap V}(y) \leq \frac{|X\cap V|+1}{2}$ and since $|X \cap V| \geq \lceil \frac{n}{4} \rceil \geq 2$, we know that one of the colors used in $X \cap V$ at y is $c(vy)$. Recall that, since $y \in D$, all edges from y to C have color $c(vy)$. This implies

$$\begin{aligned} d^c(y) &\leq \frac{|X \cap V| + 1}{2} + |G \setminus (C \cup (X \cap V) \cup \{v, x\})| \\ &\leq n' - \frac{|X \cap V| - 1}{2} - 2 \\ &< \frac{n}{2}, \end{aligned}$$

a contradiction.

Case 2. *v neither follows the colors of C nor has all the same color to C.*

Since v does not have all the same color to C, suppose v has color c to v_1 and color c' to v_2 for some $c \neq c'$. If $c \neq c(v_\ell v_1)$ and $c' \neq c(v_2v_3)$, then $C' = v_1vv_2Cv_1$ is a long proper cycle containing v so suppose, without loss of generality, that $c' = c(v_2v_3)$. If $c(vv_3) \notin \{c', c(v_3v_4)\}$, then $C' = v_2vv_3Cv_2$ is a long properly colored cycle containing v. In general, this means that whenever $c(vv_i) = c(v_iv_{i+1})$, we have $c(vv_{i+1}) \in \{c(vv_i), c(v_{i+1}v_{i+2})\}$ for $i \geq 3$. Since v does not follow the colors of C, there must exist a vertex v_j such that $c(vv_{j+1}) = c(vv_j)$. Suppose j is the smallest such index. If $j \geq 3$, then $c(vv_{j-1}) = c(v_{j-1}v_j)$ and $c(v_jv_{j+1}) = c(vv_j) = c(vv_{j+1})$ so $C' = v_{j-1}vv_{j+1}Cv_{j-1}$ is a long proper cycle containing v. Thus, we may suppose $j = 2$ meaning that $c(vv_2) = c(vv_3) = c(v_2v_3)$.

If $c(vv_1) \neq c(v_\ell v_1)$, then $C' = v_1vv_3Cv_1$ is a long proper cycle containing v so we assume $c(vv_1) = c(v_\ell v_1)$. As a mirror image to the above argument, if $c(vv_\ell) \notin \{c(v_\ell v_{\ell-1}), c(vv_1)\}$, then $C' = v_\ell vv_1Cv_\ell$ is a long proper cycle containing v. Again since v does not follow the colors of C, there exists a largest index j such that $c(vv_j) \neq c(v_{j-1}v_j)$, meaning that $c(vv_j) = c(vv_{j+1})$. If $j < \ell$, then $C' = v_jvv_{j+1}Cv_j$ is a long proper cycle containing v so suppose $j = \ell$. This means $c(vv_\ell) = c(vv_1) = c(v_\ell v_1)$. Finally, $C' = v_\ell vv_3Cv_\ell$ is a proper cycle of length at least 5 (since $|C| \geq 6$), completing the proof.

Case 3. *v follows the colors of C.*

Without loss of generality, suppose v follows the colors of C increasing. First, we show that C has DP_v. If not, there exists a chord v_iv_j of C for which $c(v_iv_j) \notin \{c(v_iv_{i+1}), c(v_jv_{j+1})\}$. Then either $C' = vv_iv_jv_{j+1}Cv_{i-1}v$ or $C'' = vv_jv_iv_{i+1}Cv_{j-1}v$ is a properly colored cycle of length at least 5 (since $|C| \geq 6$).

Let W be the set of vertices in $G \setminus C$ which do not follow C increasing. Note that $v \notin W$ and possibly $W = \emptyset$. This means that for each vertex $w \in W$, there exists an edge wv_i with $c(wv_i) \neq c(v_iv_{i+1})$. We have $c(wv_i) = c(vw)$ since otherwise either $C' = vwv_iv_{i+1}Cv_{i-1}v$ or $C'' = vwv_iv_{i+1}Cv_{i-2}v$ is a long proper cycle containing v.

We would like to show that W also has DP_v. If not, we suppose $c(w_1w_2) \notin \{c(vw_1), c(vw_2)\}$ and for notation, suppose $c(w_2v_i) = c(vw_2) \neq c(v_iv_{i+1})$. Then either $C' = vw_1w_2v_iv_{i+1}Cv_{i-1}v$ or $C'' = vw_1w_2v_iv_{i+1}Cv_{i-2}v$ is a long proper cycle containing v. Hence, W has DP_v.

We observed that $c(wv_i) = c(vw)$ whenever $c(wv_i) \neq c(v_iv_{i+1})$, so it is clear that $C \cup W$ has DP_v. By Fact 1, there exists a vertex $u \in C \cup W$ with $d^c_{C\cup W}(u) \leq \frac{|C\cup W|+1}{2}$ which means that u must have an edge of a new color (not already used on an edge incident to u in $C \cup W$) to a vertex $v' \in G \setminus (C \cup W)$ with $v' \neq v$. By the definition of W, v' must follow C increasing so if $u \in C$, this is a contradiction. This means that we may assume $u \in W$.

Since $u \in W$, there exists a vertex $v_i \in C$ such that $c(uv_i) \neq c(v_iv_{i+1})$. Suppose $c(v'u) \neq c(uv_i)$. Then at least one of $C' = v'uv_iCv_{i-1}v'$ or $C'' = v'uv_iCv_{i-2}v'$ is a properly colored cycle that is longer than C, contradicting the maximality of $|C|$. Hence, $c(v'u) = c(uv_i)$ so u has no new colors to $G \setminus (C \cup W)$, meaning that $d^c(u) < \frac{n+1}{2}$, a contradiction. This completes the proof of Theorem 5. □

The results presented in this section suffice to prove Theorem 2.

3. Proper connectivity

Our final main result provides proper connectivity in any graph with sufficient color degree.

Theorem 6. *If $n \geq 3$ and $\delta^c(G) \geq \frac{n}{2}$ then G is properly connected.*

Proof. Suppose G is an edge maximal counterexample, meaning that $\delta^c(G) \geq \frac{n}{2}$ and there exists a pair of vertices u and v which are not connected by a properly colored path but the addition of any missing edge in any color will produce such a path. Note that if G is complete, the result is immediate.

Let B be the set of vertices in $G \setminus \{u, v\}$ that are reachable from both u and v by a properly colored path. Certainly $N(u) \cap N(v) \neq \emptyset$ by the degree condition and $N(u) \cap N(v) \subseteq B$ so $B \neq \emptyset$.

We now define some notation that will be used heavily throughout the proof. Given a path $P = v_1 v_2 \ldots v_{p-1} v_p$, we define the *terminal color* of P to be the color of the edge $v_{p-1} v_p$.

Claim 1. *For each vertex $w \in N(u) \cap B$, let $c(w)$ be the color of the edge uw. Then the terminal color of every properly colored path from u to w is $c(w)$. Symmetrically, the same holds for all $w \in N(v) \cap B$.*

Proof of Claim 1. Since $w \in B$, there exists a properly colored path P from v to w and clearly the terminal color of this path must be $c(w)$ to avoid a proper uv path. Suppose there exists a proper path Q from u to w on which the terminal color is not $c(w)$. If $Q \cap P = \emptyset$, then $uQwPv$ forms a proper path from u to v so $Q \cap P \neq \emptyset$. Let x be the vertex of $Q \cap P$ which is closest to v on P and let $c(x)$ be the color of the edge from x toward v on P. Because x is an internal vertex of the path Q, there are two edges of Q containing x. At least one of these edges does not have color $c(x)$. Using this edge and the corresponding subpath of Q, there is either a properly colored path $uQxPv$ or $uwQxPv$, either case providing a contradiction. □

In general, for any vertex w, if the terminal color of every proper uw path is the same, we call that color $c(w)$.

Now suppose there is a vertex $w \in (N(u) \cap B) \setminus N(v)$. If we add the edge vw in color $c(w)$, by the maximality of $|E(G)|$, there must exist a proper uv path P using this edge. The subpath of P from u to w must then be a proper path from u to w on which the terminal color is not $c(w)$, a contradiction to Claim 1. Hence, we have shown the following fact.

Fact 2. $N(u) \cap B = N(v) \cap B$.

Next suppose there is a missing edge $wx \notin E(G)$ with $w, x \in N(u) \cap B$. If we add the edge in color $c(w)$ (or $c(x)$, it does not matter) then there must exist a proper path from u to v using that edge. This would mean that there was a proper path from either u or v to w of which the terminal color is not $c(w)$, a contradiction. Hence, by the maximality of G, all edges must be present within $N(u) \cap B$ and certainly $N(u) \cap B$ has DP_u.

Let $B_1 = B \cap N(u) = B \cap N(v)$ and let $B_2 = B \setminus B_1$. Suppose there exists $x \in B_2$ and $w \in B_1$ with $xw \notin E(G)$. If we add the edge xw in color $c(w)$, there must exist a proper uv path but this means that there must again be a path from either u or v to w of which the terminal color is not $c(w)$, a contradiction to Claim 1. This means that all edges must be present between B_1 and B_2.

We would now like to show that any shortest proper path P from $x \in B_2$ to v uses exactly one vertex of B_1. Orient P from x toward v. For a vertex $y \in P$, let $c^-(y)$ be the color of the edge coming in to y on P and let $c^+(y)$ be the color of the edge going out of y on P. Consider the first vertex $w \in P \cap B_1$. If $c^-(w) \neq c(w)$ then there is a properly colored path from v to w with terminal color not equal to $c(w)$, a contradiction to Claim 1. This means that $c^-(w) = c(w)$ and $c^+(w) \neq c(w)$. Now let w' be the last internal vertex of P. Certainly $w' \in B_1$ and $c^-(w') \neq c(w')$. If we let Q be the subpath of P from w to w', then $uwQw'v$ is a proper path from u to v not using the edge ux, a contradiction. This implies the following fact.

Fact 3. *Any shortest proper path P from a vertex $x \in B_2$ to v (or u) must intersect B_1 in exactly one vertex.*

Now let x be a vertex in B_2 and suppose x has at least 2 edges to vertices $w_1, w_2 \in B_1$ which do not have colors $c(w_1)$ or $c(w_2)$ respectively. Certainly the colors of xw_1 and xw_2 must be the same (call it $c(x)$) since otherwise the path uw_1xw_2v would be proper. Now add the edge ux in color $c(x)$. By the maximality of G, there must be a proper uv path using this edge. This means there is a proper path from v to x of which the terminal color is not $c(x)$. Consider the shortest such path P.

Since P uses only one vertex of B_1 by Fact 3, we know that one of w_1 or w_2 is not in P, suppose w_i is such a vertex (where $i \in \{1, 2\}$). Now the path uw_ixPv is a proper uv path avoiding the edge ux, a contradiction. This means that, for each vertex $x \in B_2$, x has at most one edge to a vertex $w \in B_1$ which has color $c(x) \neq c(w)$. Call such an edge *special*.

Now suppose x has a special edge to a vertex $w \in B_1$. If we again add the edge ux in color $c(x)$, there must be a proper path P from x to v on which the first edge does not have color $c(x)$ and if we choose the shortest such path, by Fact 3, this must intersect B_1 in exactly one vertex. If $B_1 \cap P$ is not w, then $uwxPv$ is a proper uv path in G so we know that $\{w\} = B_1 \cap P$. This means that if a vertex $w \in B_1$ has a special edge to B_2, then w must have at least 2 special edges to B_2.

Let $C \subseteq B_1$ be the set of vertices with no special edges. Since each vertex of B_2 has at most one special edge and each vertex of $B_1 \setminus C$ has at least two, we see that

$$|B_2| \geq 2|B_1 \setminus C|. \tag{1}$$

By the color degree assumption, $d^c(u) \geq \frac{n}{2}$ so $|G \setminus B_2| \geq \frac{n}{2} + 2$ which means that $|B_2| \leq \frac{n}{2} - 2$. By Eq. (1), we see that $|B_1 \setminus C| \leq \frac{n}{4} - 1$.

If $C = \emptyset$, that would mean that each of u and v must have at least $\frac{n}{4} + 1$ neighbors each outside of B. Let U be the vertices of $N(u) \setminus B$ and $V = N(v) \setminus B$. This means that there are at least $|U \cup V| \geq \frac{n}{2} + 2$ vertices outside $B \cup \{u, v\}$. Recall that (by the definition of B) each vertex $w \in B_1$ has only edges of color $c(w)$ to vertices outside B so $|B| \geq \frac{n}{2}$. This is a contradiction since we have shown $n = |B| + |U \cup V| + 2 \geq n + 4$, meaning that $C \neq \emptyset$.

If we let z be a vertex in C, then z has only edges of color $c(z)$ to vertices outside B_1. By the degree condition, since $C \neq \emptyset$, we know that $|B_1| \geq \frac{n}{2}$. More specifically, since all edges between vertices $y_1, y_2 \in C$ must have either color $c(y_1)$ or $c(y_2)$, we may consider a vertex of smallest color degree within C to see that

$$\frac{|C|+1}{2} + |B_1 \setminus C| \geq \frac{n}{2}. \tag{2}$$

By Eqs. (1) and (2), we see that

$$\frac{|B_2|}{2} \geq |B_1 \setminus C| \geq \frac{n-|C|-1}{2}$$

which implies $|B_2| + |C| \geq n - 1$, clearly a contradiction. This completes the proof of Theorem 6. $\square$

In order to demonstrate the sharpness of this result, consider the graph G_2 from Construction 1. This graph has $\delta^c(G_2) = \frac{n-1}{2}$ but there is no proper path from x to y. For the case where n is even, the graph G_2' has $\delta^c(G_2') = \frac{n}{2} - 1$ and still no proper path from x to y.

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